Utility-based Decision-making in Distributed Systems Modelling

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This technical report provides proofs for the paper ‘Utility-based Decision-making in Distributed Systems Modelling [Extended Abstract]’ [1]. This report also provides proofs for the paper ‘Trust Domains: An Algebraic, Logical, and Utility-theoretic Approach’ [2].

1 Proofs for ‘Utility-based Decision-making in Distributed Systems Modelling [Extended Abstract]’

A key technical property, which underpins much of the remaining work, is the following: if bisimilar contexts are substituted into each other, then the result is bisimilar. We provide a detailed sketch of the (very long) proof that suitably underpins the result.

Proposition 1. If \(E \sim G\) and \(F \sim H\), then \(E(F) \sim G(H)\).

Proof.

The bisimulation relation \(\sim\) is the largest bisimulation relation, and contains all other bisimulation relations. In order to show that \(E(F) \sim G(H)\) it is sufficient, therefore, to define a relation \(\mathcal{R}\), where \(E(F) \mathcal{R} G(H)\), for which the required substitution property holds, and show that the relation \(\mathcal{R}\) is a bisimulation.

Let \(\mathcal{R} = \{(E(F), G(H)) | E \sim G \text{ and } F \sim H\} \cup \sim\). The relation is a bisimulation if and only if the following holds. For all \(T, c, T', I', C_1 \sim C_3,\) and \(C_2 \sim C_4,\) if \(T, E(F) \xrightarrow{C_2 \sim C_1} T', I'\), then there exists some \(J'\) such that \(T, G(H) \xrightarrow{C_4 \sim C_3} T', J',\) where \((I', J') \in \mathcal{R}\); and, for all \(T, c, T', J', C_1 \sim C_3,\) and \(C_2 \sim C_4,\) if \(T, G(H) \xrightarrow{C_4 \sim C_3} T', J',\) then there exists some \(I'\) such that \(T, E(F) \xrightarrow{C_2 \sim C_1} T', I',\) where \((I', J') \in \mathcal{R}\).

All processes are defined by a finite number of applications of the operators of the language. We proceed by induction on the derivation of this structure according to the rules of the operational semantics.

Consider the case where \(T, E(F) \xrightarrow{C_2 \sim C_1} T, I'\). We prove that there exists some \(J'\) such that \(T, G(H) \xrightarrow{C_4 \sim C_3} T, J'\) by induction on the structures of \(E, G, F,\) and \(H\), and over the (process) structures of \(C_1, C_3, C_2,\) and \(C_4,\) in that order. Here the induction is on the number of operators in a process term.

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Consider the case of this nested induction where \( E = 1 \), \( G = G_1 \times G_2 \), \( F = 1 \), and \( H = 1 \), and where \( C_1 = e, [ ] \), \( C_3 = e, [ ] \), \( C_2 = e, 1 \), and \( C_4 = e, 1 \).

In SCCS, if \( 1 \sim G_1 \times G_2 \), then \( G_1 \) and \( G_2 \) would necessarily be bisimilar to \( 1 \), but here that is not the case. Consider the process \( G_1 = 1 +_u (a : 1) \), where

\[
u(C) = \begin{cases} 
  n & \text{if there exists } C' : OCont, R \text{ where } C = C'(R, 1 \times G_2) \\
  0 & \text{otherwise,}
\end{cases}
\]

for some \( n > 0 \).

We then have that \((1 +_u (a : 1)) \times G_2 \sim 1\); a sketch of the proof follows below.

Consider some contexts \( C_5 \sim C_7 \) and \( C_6 \sim C_8 \), and resources \( R = R_1 \circ R_2 \). We then have that \( \nu(C_5(R, 1 \times G_2)) = n \) and that \( \nu(C_5(R, a : 1 \times G_2)) = 0 \), for all \( C_5 \). By the \((\text{Sum})\) rule, we have that

\[
R_1, 1 +_u (a : 1) \xrightarrow{C_2} R_1, 1 \text{ and that } R_1, 1 +_u (a : 1) \xrightarrow{C_5(R_1, [ ] \times G_2)} .
\]

Note, however, that \( 1 +_u (a : 1) \) is not bisimilar to \( 1 \); for the empty outer context \( e, [ ] \), the options \( 1 \) and \( a : 1 \) are valued equally. In this case, either can be chosen, and hence the process \( 1 +_u (a : 1) \) can perform an \( a \) action transition while the process \( 1 \) cannot.

As a result, bisimulation does not work component-by-component (solely because of utility-modulated choice). Note, however, that following a utility-modulated choice by \( 1 +_u (a : 1) \), the resulting process \( 1 \) is bisimilar to \( 1 \). As the reduction of utility-modulated choices applies to strictly simpler terms, we can apply the induction hypothesis and show that substituted processes are bisimilar, and hence the choices that are formed from them are also bisimilar.

This argument extends to more complex cases, where the subcomponents of a product are not merely built from tick processes. There, cost moderated choices as subcomponents will also eventually reduce to a product subcomponent to which the induction hypothesis can be applied.

The proof sketched above makes use of an eight-fold nested induction, which has a very large number of cases. Unfortunately, the complexity of this nested induction precludes an exhaustive presentation in a reasonably concise form. However, it is easy to see that the difficult cases in the induction arise solely from use of utility-modulated choices and, since these always reduce to simpler processes to which the induction hypothesis can be applied, the key cases are all similar to the one described above.

The remaining cases are routine. \( \square \)

Note that this argument would not be applicable in the presence of general (guarded) fixed points; our contextual calculus must be a finite modelling framework.

With this result, we can obtain a key property for reasoning compositionally, that bisimulation is a congruence.

**Theorem 1** (Bisimulation Congruence). The relation \( \sim \) is a congruence. It is reflexive, symmetric, and transitive, and, for all \( a, E, F, G \), with \( E \sim F \), and all families \( (E_i)_{i \in I}, (F_i)_{i \in I} \) with \( E_i \sim F_i \), for all \( i \in I \), \( a : E \sim a : F, E \times G \sim F \times G, \) and \( \sum_{i \in I} E_i \sim \sum_{i \in I} F_i \).

**Proof.** By induction on the structure of bisimulations. We give illustrative cases.

1. Reflexive. Let \( R = \{ (E, E) \} \). As the evolution of each of the process \( E \) is decorated with different outer and inner contexts \( (C_1 \) and \( C_2 \), and \( C_3 \) and \( C_4 \), respectively), we do not immediately have reflexivity. We prove this property by induction on the derivation of \( R, E \xrightarrow{C_2, a} E' \).
The relation \( \mathcal{R} \) is a bisimulation if and only if, for all \( E \) and \( F \) such that \( E \mathcal{R} F \), the following holds: for all \( R, R', E', a, C_1 \sim C_3 \), and \( C_2 \sim C_4 \), if \( R, E \xrightarrow{C_1} C_3 \), \( R' \), \( E' \), then there exists some \( F' \) such that \( R, F \xrightarrow{C_1} C_3 \), \( R', F' \), where \( E' \mathcal{R} F' \); and, for all \( R, R', F', a, C_1 \sim C_3 \), and \( C_2 \sim C_4 \), if \( R, F \xrightarrow{C_1} C_3 \), \( R', F' \), then there exists some \( E' \) such that \( R, E \xrightarrow{C_1} C_3 \), \( R', E' \), where \( E' \mathcal{R} F' \).

- Consider some \( E \mathcal{R} F, R', E', a, C_1 \sim C_3 \), and \( C_2 \sim C_4 \) such that \( R, E \xrightarrow{C_1} C_3 \), \( R', E' \). By the definition of \( \mathcal{R} \), we know that \( F = E \). We then prove that there exists some \( E'' \), such that \( R, E \xrightarrow{C_1} C_3 \), \( R', E'' \) and \( E' \mathcal{R} E'' \), by induction over the derivation of \( R, E \xrightarrow{C_1} C_3 \), \( R', E'' \).

- Consider some \( E \mathcal{R} F, R', F', a, C_1 \sim C_3 \), and \( C_2 \sim C_4 \) such that \( R, F \xrightarrow{C_1} C_3 \), \( R', F' \). By the definition of \( \mathcal{R} \), we know that \( F = E \) and \( F' = E'' \). We then prove that there exists some \( E' \), such that \( R, E \xrightarrow{C_1} C_3 \), \( R', E' \) and \( E' \mathcal{R} E'' \), by induction over the derivation of \( R, E \xrightarrow{C_1} C_3 \), \( R', E'' \).

Hence \( \mathcal{R} \) is closed and a bisimulation.

2. Symmetric. Let \( \mathcal{R} = \{(F, E) \mid E \sim F \} \cup \sim \). If \( R, F \xrightarrow{C_1} C_3 \), \( S, F', C_1 \sim C_3 \), and \( C_2 \sim C_4 \), then we need to show that \( R, E \xrightarrow{C_1} C_3 \), \( S, E' \), where \( F' \mathcal{R} E' \). As \( E \sim F \), by the definition of bisimulation, we have that, if \( C_5 \sim C_7, C_6 \sim C_8 \), and \( R, F \xrightarrow{C_1} C_3 \), \( S, F' \), then \( R, E \xrightarrow{C_1} C_3 \), \( S, E' \), where \( E' \sim F' \).

Let \( C_1 = C_5, C_2 = C_6, C_3 = C_7, C_4 = C_8 \). We then have that \( R, E \xrightarrow{C_1} C_3 \), \( S, E' \). As \( E' \sim F' \), we have that \( F' \mathcal{R} E' \). The other case is similar. Hence \( \mathcal{R} \) is closed and a bisimulation.

3. Transitive. Let \( \mathcal{R} = \{(E, G) \mid E \sim F \land F \sim G \} \). If \( R, E \xrightarrow{C_1} C_3 \), \( S, E', C_1 \sim C_3 \), and \( C_2 \sim C_4 \), then we need to show that \( R, G \xrightarrow{C_1} C_3 \), \( S, G' \), where \( E' \mathcal{R} G' \). By the definition of bisimulation, as \( R, E \xrightarrow{C_1} C_3 \), \( S, E' \), we have that \( R, F \xrightarrow{C_1} C_3 \), \( S, F' \), where \( E' \sim F' \), and, similarly, as \( R, F \xrightarrow{C_1} C_3 \), \( S, F' \), we have that \( R, G \xrightarrow{C_1} C_3 \), \( S, G' \), where \( F' \sim G' \). We then have that \( E' \mathcal{R} G' \). The other case is similar. Hence \( \mathcal{R} \) is closed and a bisimulation.

4. Let \( \mathcal{R} = \{a \mid a \in \{E \sim F \} \} \). If \( R, a : E \xrightarrow{C_1} C_3 \), \( S, E, C_1 \sim C_3 \), and \( C_2 \sim C_4 \), then we need to prove that \( R, a : F \xrightarrow{C_1} C_3 \), \( S, F \), where \( E \mathcal{R} F \). The only applicable reduction rule for \( a : E \) is the (Prefix) rule. By this rule, which disregards \( C_3 \) and \( C_4 \), we can show that \( R, a : F \xrightarrow{C_1} C_3 \), \( S, F \). As \( E \sim F \), we have that \( E \mathcal{R} F \). The other case is similar. Hence \( \mathcal{R} \) is closed and a bisimulation.

5. Let \( \mathcal{R} = \{(E \xrightarrow{u} F \xrightarrow{u} G) \mid E \sim F \} \). If \( R, E \xrightarrow{C_1} C_3 \), \( R', H', C_1 \sim C_3 \), and \( C_2 \sim C_4 \), then we need to prove that \( R, F \xrightarrow{C_1} C_3 \), \( R', I' \), where \( H' \mathcal{R} I' \). By the (Send) rule, we know
that either \( R, E \xrightarrow{C_2} R', H' \) and \( u(C_1(R, G(C_2))) \leq u(C_1(R, E(C_2))) \), or \( R, G \xrightarrow{C_4} R', H' \) and \( u(C_1(R, E(C_2))) \leq u(C_1(R, G(C_2))) \), where \( C_5 = C_1(e, G(C_2) + u[ ]) \) and \( C_6 = C_1(e, E(C_2) + u[ ]) \).

- Consider the former case. Let \( C_7 = C_3(e, G(C_4) + u[ ]) \). As \( E \sim F \) and \( C_5 \sim C_7 \) (by Proposition 1), by the definition of bisimulation, we have that \( R, F \xrightarrow{C_7} R', I' \), where \( H' \sim I' \). As \( C_1(R, E(C_2)) \sim C_3(R, F(C_4)) \) and \( C_1(R, G(C_2)) \sim C_3(R, G(C_4)) \) (by Proposition 1), then, by Definition 1, we have that \( u(C_1(R, E(C_2))) = u(C_3(R, F(C_4))) \) and \( u(C_1(R, G(C_2))) = u(C_3(R, G(C_4))) \). Hence, we have that \( u(C_3(R, G(C_4))) \leq u(C_3(R, F(C_4))) \). By the \((\text{Ssm})\) rule, we can show that \( R, F \xrightarrow{C_4} R', I' \). As \( H' \sim I' \), we have that \( H' \Downarrow R' \).

- Consider the latter case. Let \( C_8 = C_3(e, F(C_4) + u[ ]) \). As \( G \sim F \) and \( C_6 \sim C_8 \) (by Proposition 1), by the definition of bisimulation, we have that \( R, G \xrightarrow{C_6} R', I' \), where \( H' \sim I' \). As \( C_1(R, E(C_2)) \sim C_3(R, F(C_4)) \) and \( C_1(R, G(C_2)) \sim C_3(R, G(C_4)) \) (by Proposition 1), then, by Definition 1, we have that \( u(C_1(R, E(C_2))) = u(C_3(R, F(C_4))) \) and \( u(C_1(R, G(C_2))) = u(C_3(R, G(C_4))) \). Hence, we have that \( u(C_3(R, F(C_4))) \leq u(C_3(R, G(C_4))) \). By the \((\text{Ssm})\) rule, we can show that \( R, F \xrightarrow{C_4} R', I' \). As \( H' \sim I' \), we have that \( H' \Downarrow R' \).

The other case is similar. Hence \( R \) is closed, and a bisimulation.

6. Let \( R = \{(E \times G, F \times G) \mid E \sim F \} \). If \( R \circ S, E \times G \xrightarrow{C_2} R' \circ S', E' \times G' \), \( C_1 \sim C_3 \) and \( C_2 \sim C_4 \), then we need to prove that \( R \circ S, F \times G \xrightarrow{C_1} R' \circ S', F' \times G'' \), where \( E' \sim G' \Downarrow R F' \times G'' \). By the \((\text{Proo})\) rule, we have that \( R, E \xrightarrow{C_2} R', E' \) and \( S, G \xrightarrow{C_2} S', G' \), where \( C_5 = C_1((S, F(C_2)) \times [ ]) \) and \( C_6 = C_1((R, E(C_2)) \times [ ]) \). Let \( C_7 = (C_3(S, G(C_4)) \times [ ]) \) and \( C_8 = (C_3(R, F(C_4)) \times [ ]) \). By Proposition 1 and \( R \), we have that \( C_5 \sim C_7 \) and \( C_6 \sim C_8 \). By the definition of bisimulation, we have that \( R, F \xrightarrow{C_4} R', F' \) and \( S, G \xrightarrow{C_4} S', G'' \), where \( E' \sim F' \) and \( G' \sim G'' \). We can then use the \((\text{Proo})\) rule to show that \( R \circ S, F \times G \xrightarrow{C_1} R' \circ S', F' \times G'' \), where \( E' \times G' \Downarrow R F' \times G'' \). The other cases are similar. Hence \( R \) is closed and a bisimulation.

\( \square \)

In [1], there was a slight mis-statement of the second part of Definition 2. We provide a correct statement below.

**Definition 2.** The set of utilities, \( U \), is (algebraically) accordant if it respects bisimilarity and, for all \( u, v \in U \), all \( C, C_1, C_2, C_3, C_4 : \text{Cont} \), all \( E, F, G \in P\text{Cont} \), and \( R \in R \),

2. If \( u(C(R, F)) \leq u(C(R, G)) \), then \( u(C(R, G)) \leq u(C(R, E) + u[ F]) \).

In order to reason equationally about processes, it is also useful to establish various algebraic properties concerning parallel composition and choice. We derive these properties for our calculus below. We use the binary version of sum here in order to aid comprehension, but finite choices between sets of processes work straightforwardly.
Proposition 2 (Algebraic Properties). If $U$ is accordant, then:

$$
\begin{align*}
1 & \quad E + u F \sim F + u E & 5 & \quad E \times 1 \sim E \\
2 & \quad E + u (F + u G) \sim (E + u F) + u G & 6 & \quad E \times F \sim F \times E \\
3 & \quad E + u 0 \sim E & 7 & \quad E \times (F \times G) \sim (E \times F) \times G \\
4 & \quad E \times 0 \sim 0 & 8 & \quad (E + u F) \times G \sim E \times G + u F \times G.
\end{align*}
$$

This result provides algebraic properties that are standard in well-formed process calculi. Note that Definitions 2.1 and 2.2 are used to prove Property 2. Definition 2.3 is used to prove Property 3. Properties 1 and 4-8 do not require any of the accordance definitions. When we take $u = 0$, then the accordance properties hold vacuously, and we obtain the behaviour of a traditional process calculus with non-deterministic choice that dis-regards the context.

Proof.

1. Let $\mathcal{R} = \{(E + u F, F + u E) \mid E, F : PCont\} \cup \sim$. The relation $\mathcal{R}$ is a bisimulation if and only if, for all $E_1$ and $E_2$ such that $E_1 \mathcal{R} E_2$, the following holds: for all $R$, $R'$, $E_1'$, $a$, $C_1 \sim C_3$, and $C_2 \sim C_4$, if $R, E_1 \xrightarrow{C_2} \mathcal{R}, E_1'$, then there exists some $E_2'$ such that $R, E_2 \xrightarrow{C_4} \mathcal{R}, E_2'$, where $E_1' \mathcal{R} E_2'$; and, for all $R$, $R'$, $E_2'$, $a$, $C_1 \sim C_3$, and $C_2 \sim C_4$, if $R, E_2 \xrightarrow{C_4} \mathcal{R}, E_2'$, then there exists some $E_1'$ such that $R, E_1 \xrightarrow{C_2} \mathcal{R}, E_1'$, where $E_1' \mathcal{R} E_2'$.

- Consider some $E_1 \mathcal{R} E_2$, $R'$, $E_1'$, $a$, $C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, E_1 \xrightarrow{C_2} \mathcal{R}, E_1'$. Consider the case in which $E_1 \sim E_2$. Then, by the definition of bisimulation, we have that there exists some $E_2'$ such that $R, E_2 \xrightarrow{C_4} \mathcal{R}, E_2'$, where $E_1' \sim E_2'$, and hence $E_1' \mathcal{R} E_2'$. Consider the case in which $E_1 = E + u F$ and $E_2 = F + u E$. By the $(\Sigma)$ rule, we know that either $R, E \xrightarrow{C_2} \mathcal{R}, E_1'$ and $u(C_1(R, F(C_2))) \leq u(C_1(R, E(C_2)))$, or $R, F \xrightarrow{C_4} \mathcal{R}, E_1'$ and $u(C_1(R, E(C_2))) \leq u(C_1(R, F(C_2)))$, where $C_5 = C_1(e, F(C_2) + u [\ ])$ and $C_6 = C_1(e, F(C_2) + u [\ ])$.

- Consider the former case. Let $C_7 = C_3(e, F(C_4) + u [\ ])$ and $C_8 = C_7$ (by Proposition 1), then, by the definition of bisimulation, we have that $R, F \xrightarrow{C_4} \mathcal{R}, E_2'$, where $E_1' \sim E_2'$. As $C_1(R, E(C_2)) \sim C_3(R, E(C_4))$ and $C_1(R, F(C_2)) \sim C_3(R, F(C_4))$, we have that $C_5 \sim C_7$ (by Proposition 1), then, by Definition 1, we have that $u(C_1(R, E(C_2))) = u(C_3(R, E(C_4)))$ and $u(C_1(R, F(C_2))) = u(C_3(R, F(C_4)))$. Hence we have that $u(C_3(R, F(C_4))) \leq u(C_3(R, E(C_4)))$. By the $(\Sigma)$ rule, we can show that $R, F + u E \xrightarrow{C_4} \mathcal{R}, E_2'$. As $E_1' \sim E_2'$, we have that $E_1' \mathcal{R} E_2'$.

- Consider the latter case. Let $C_8 = C_3(e, E(C_4) + u [\ ])$ and $C_6 = C_8$ (by Proposition 1), then, by the definition of bisimulation, we have that $R, F \xrightarrow{C_4} \mathcal{R}, E_2'$, where $E_1' \sim E_2'$. As $C_1(R, E(C_2)) \sim C_3(R, E(C_4))$ and $C_1(R, F(C_2)) \sim C_3(R, F(C_4))$, we have that $C_5 \sim C_7$ (by Proposition 1), then, by Definition 1, we have that $u(C_1(R, E(C_2))) = u(C_3(R, E(C_4)))$ and $u(C_1(R, F(C_2))) = u(C_3(R, F(C_4)))$. Hence we have that $u(C_3(R, E(C_4))) \leq u(C_3(R, F(C_4)))$.
2. Let \( R, S \sim (C, C) \). Then, by the definition of bisimulation, we have that \( R, S \sim (C, C) \). Consider the latter case. Let \( C_7 = C_1(e, F(C_2) + u [ ]) \). As \( C_5 \sim C_7 \), by Proposition 1, we have that \( R, F \rightarrow C_7 \). As \( C_6 \sim C_8 \), by Proposition 1, we have that \( R, F \rightarrow C_8 \), where \( E_1 \sim E_2 \). As \( C_1(R, E(C_2)) \sim C_3(R, E(C_4)) \), and \( C_1(R, F(C_2)) \sim C_3(R, F(C_4)) \), we have that \( u(C_1(R, E(C_2))) = u(C_3(R, E(C_4))) \). Hence we have that \( u(C_1(R, F(C_2))) \leq u(C_1(R, F(C_2))) \). By the \((\text{Sum})\) rule, we can show that \( R, F \rightarrow C_6 \), where \( E_1 \sim E_2 \). As \( E_1 \sim E_2 \), we have that \( E_1 \rightarrow E_2 \). We consider the former case. Let \( C_8 = C_1(e, E(C_2) + u [ ]) \). As \( C_6 \sim C_8 \), by Proposition 1, we have that \( R, F \rightarrow C_8 \), where \( E_1 \sim E_2 \). As \( C_1(R, E(C_2)) \sim C_3(R, E(C_4)) \), and \( C_1(R, F(C_2)) \sim C_3(R, F(C_4)) \), we have that \( u(C_1(R, E(C_2))) = u(C_3(R, E(C_4))) \). Hence we have that \( u(C_1(R, F(C_2))) \leq u(C_1(R, F(C_2))) \). By the \((\text{Sum})\) rule, we can show that \( R, F \rightarrow C_6 \), where \( E_1 \sim E_2 \), we have that \( E_1 \rightarrow E_2 \). Hence \( R \) is closed and a bisimulation.

Let \( R = \{(E + u(F + u(G)), (E + u(F) + u(G)) : \text{PCont} \} \)

Let \( R, S \sim (C, C) \), then we need to show that \( R, S \sim (C, C) \). Consider the latter case. Let \( C_7 = C_1(e, F(C_2) + u [ ]) \). As \( C_5 \sim C_7 \), by Proposition 1, we have that \( R, F \rightarrow C_7 \). As \( C_6 \sim C_8 \), by Proposition 1, we have that \( R, F \rightarrow C_8 \), where \( E_1 \sim E_2 \). As \( C_1(R, E(C_2)) \sim C_3(R, E(C_4)) \), and \( C_1(R, F(C_2)) \sim C_3(R, F(C_4)) \), we have that \( u(C_1(R, E(C_2))) = u(C_3(R, E(C_4))) \). Hence we have that \( u(C_1(R, F(C_2))) \leq u(C_1(R, F(C_2))) \). By the \((\text{Sum})\) rule, we can show that \( R, F \rightarrow C_6 \), where \( E_1 \sim E_2 \), we have that \( E_1 \rightarrow E_2 \). Hence \( R \) is closed and a bisimulation.

Let \( C_9 = C_3(e, G(C_4) + u [ ]) \), \( C_{10} = C_3(e, E + u(F(C_4) + u [ ]) \), \( C_{11} = C_9(e, F(C_4) + u [ ]) \), \( C_{12} = C_9(e, E(C_2) + u [ ]) \), \( C_{13} = C_3(e, F + u(G(C_4) + u [ ]) \), \( C_{14} = C_9(e, E(C_4) + u [ ]) \), \( C_{15} = C_9(e, G(C_4) + u [ ]) \), \( C_{16} = C_9(e, F(C_4) + u [ ]) \).
3. Let $\mathcal{R} = \{(E \leftrightarrow_0 \top, E) \mid E \in \text{PCont}) \cup \sim\}$. If $R, E \leftrightarrow_0 \top \xrightarrow{C_1} R', E'$, $C_1 \sim C_3$, and $C_2 \sim C_4$, then we need to show that $R, E \xrightarrow{C_1} S, E'$, where $E' \sim E'$. By the $(\sim_{\text{Sm}})$ rule, we know that either $R, E \xrightarrow{C_3} R', E'$ and $u(C_1(R, 0(C_2))) \leq u(C_1(R, E(C_2)))$, or $R, 0 \xrightarrow{C_6} R', E'$ and $u(C_1(R, E(C_2))) \leq u(C_1(R, 0(C_2)))$. Hence we can make the second case. Let $C_7 = C_3(e, 0(C_2) + \top]$ and $C_8 = C_1(e, E(C_2) + \top]$. By $\mathcal{R}$ and Proposition 1, we have that $C_5 \sim C_7 \sim C_3$. By the definition of bisimulation, we have that $R, E \xrightarrow{C_4} R', E'$. As $E' \sim E'$ we have that $E' \sim E'$. The other case is similar. Hence $\mathcal{R}$ is closed and a bisimulation.

4. Let $\mathcal{R} = \{(E \times \emptyset, \emptyset) \mid E \in \text{PCont})\}$. By the operational semantics, we have that $\emptyset$ can make
neither action nor weighted transitions. The only applicable rule to $E \times 0$ is (PCont). This requires, as sub-derivations, that $0$ make a transition, which is impossible. Hence $E \times 0$ can also make no transitions, and is bisimilar to $0$.

5. Let $\mathcal{R} = \{(E \times 1, F) \mid E \sim F\}$ If $R, E \times 1 \xrightarrow{c_1} S, E', C_1 \sim C_3$ and $C_2 \sim C_4$, then we need to show that $R, F \xrightarrow{c_2} S, F'$, where $E' \not\sim F'$. Let $C_5 = C_1((e, 1(C_2)) \times [])$. By the (Pro) rule, we have that $R, E \xrightarrow{c_2} S, E'$. By $\mathcal{R}$ and Proposition 1 we then have that $C_3 \sim C_5 \sim C_1$. We can then show that $R, E \xrightarrow{c_2} S, F'$ where $E' \sim F'$. We then have that $E' \times 1 \not\sim F'$. The other case is similar. Hence $\mathcal{R}$ is closed and a bisimulation.

6. Let $\mathcal{R} = \{(E \times F, F \times E) \mid E, F : PCont\}$ If $R \circ S, E \times F \xrightarrow{c_2} \text{ab} \rightarrow \rightarrow \mathcal{R} \circ S', E' \times F', C_1 \sim C_3$ and $C_2 \sim C_4$, then we need to show that $R \circ S, F \times E \xrightarrow{c_4} \text{ab} \rightarrow \rightarrow S', \mathcal{R} \circ F'' \times E''$. Let $C_5 = C_1((S, F(C_2)) \times [])$ and $C_6 = C_1((R, E(C_2)) \times [])$. By the (Pro) rule, we have that $R, E \xrightarrow{c_2} S, E'$ and $S, F \xrightarrow{c_4} S', F'$. Let $C_7 = C_3((S, F(C_4)) \times [])$ and $C_8 = C_3((R, E(C_4)) \times [])$. By $\mathcal{R}$ and Proposition 1, we have that $C_5 \sim C_7$ and $C_6 \sim C_8$. By the definition of bisimulation, we have that $R, E \xrightarrow{c_2} \text{ab} \rightarrow \rightarrow R \circ S', \mathcal{R} \circ S' \times F''$. Then, by the (Pro) rule, we have that $R \circ S, F \times E \xrightarrow{c_4} \text{ab} \rightarrow \rightarrow R \circ S', F' \times E'$, where $(E' \times F', F' \times E') \in \mathcal{R}$. As $E' \sim E''$ and $F' \sim F''$ we have that $E' \times F' \not\sim \mathcal{R} F'' \times E''$. The other case is similar. Hence $\mathcal{R}$ is closed and a bisimulation.

7. Let $\mathcal{R} = \{(E \times (F \times G), (E \times F) \times G) \mid E, F, G : PCont\}$. If $R \circ S \circ T, E \times (F \times G) \xrightarrow{c_2} \text{abc} \rightarrow \rightarrow R' \circ S' \circ T', E' \times (F' \times G')$, $C_1 \sim C_3$, and $C_2 \sim C_4$, then we need to show that $R \circ S \circ T, (E \times F) \times G \xrightarrow{c_2} \text{abc} \rightarrow \rightarrow R' \circ S' \circ T', (E' \times F') \times G'$, where $E' \times (F' \times G') \not\sim \mathcal{R} (E'' \times F'') \times G''$. Let $C_5 = C_1((S \circ T, F \times G(C_2)) \times [])$ and $C_6 = C_1((R, E(C_2)) \times [])$. By the (Pro) rule, we have that $R, E \xrightarrow{c_2} R', E'$ and that $S \circ T, F \times G \xrightarrow{c_2} S' \circ T', F' \times G'$. Let $C_7 = C_{10}((T, G(C_2)) \times [])$ and $C_8 = C_6((S, F(C_2)) \times [])$. By the (Pro) rule, we have that $S, F \xrightarrow{c_2} S', F'$ and that $T, G \xrightarrow{c_2} T', G'$. Let $C_9 = C_{10}((T, G(C_4)) \times [])$, $C_{11} = C_9((S, F(C_4)) \times [])$, $C_{12} = C_9((R, E(C_4)) \times [])$, and $C_5 \sim C_10$. By the definition of bisimulation, we have that $R, E \xrightarrow{c_2} R', E'$, $S, F \xrightarrow{c_2} S', F'$, and $T, G \xrightarrow{c_2} T', G'$, where $E' \sim E''$, $F' \sim F''$, and $G' \sim G''$. Then, by the (Pro) rule, we have that $R \circ S, E \times F \xrightarrow{c_2} \text{ab} \rightarrow \rightarrow R' \circ S', E'' \times F''$, and that $R \circ S \circ T, (E \times F) \times G \xrightarrow{c_2} \text{abc} \rightarrow \rightarrow R' \circ S' \circ T', (E'' \times F'') \times G''$. As $E' \sim E''$, $F' \sim F''$, and $G' \sim G''$, we have that $E' \times (F' \times G') \not\sim \mathcal{R} (E'' \times F'') \times G''$. The other case is similar. Hence $\mathcal{R}$ is closed and a bisimulation.

8. Let $\mathcal{R} = \{((E + F) \times G, E \times G + F \times G) \mid E, F : PCont$ and $G : PCont\}$. If
Proof. By induction over the derivation of \( C_1 \vdash C_2 \phi \).

- Case \( C_1 \vdash C_2 \top \). By the definition of \( \top \), we have that, if \( C \sim C' \) and \( C \in \mathcal{V}(p) \), then \( C' \in \mathcal{V}(p) \). By Proposition 1, we have that \( C_2(C_1) \sim C_4(C_3) \), and hence \( C_4(C_3) \in \mathcal{V}(p) \).

- Case \( C_1 \vdash C_2 \bot \). As the premisses assume \( C_1 \vdash C_2 \phi \), we have a contradiction and can disregard this case.

The other case is similar. Hence \( \mathcal{R} \) is closed and a bisimulation.

\[\Box\]

**Theorem 2.** If \( C_1 \vdash C_2 \phi \), and \( C_1 \sim C_3 \), and \( C_2 \sim C_4 \), then \( C_3 \vdash C_4 \phi \).

**Proof.** By induction over the derivation of \( C_1 \vdash C_2 \phi \).

- Case \( C_1 \vdash C_2 \phi \). The definition of \( \phi \), we have that, if \( C \sim C' \) and \( C \in \mathcal{V}(p) \), then \( C' \in \mathcal{V}(p) \). By Proposition 1, we have that \( C_2(C_1) \sim C_4(C_3) \), and hence \( C_4(C_3) \in \mathcal{V}(p) \).

The other case is similar. Hence \( \mathcal{R} \) is closed and a bisimulation.
• Case $C_1 \Vdash_{C_2} \top$. We straightforwardly have that $C_3 \Vdash_{C_4} \top$.

• Case $C_1 \Vdash_{C_2} \phi \land \psi$. By the induction hypothesis, we know that $C_3 \Vdash_{C_4} \phi$ and $C_3 \Vdash_{C_4} \psi$. Hence we have that $C_3 \Vdash_{C_4} \phi \land \psi$.

• Case $C \Vdash_{C'} \phi \lor \psi$. By the induction hypothesis, we know that $C_3 \Vdash_{C_4} \phi$ or $C_3 \Vdash_{C_4} \psi$. Hence we have that $C_3 \Vdash_{C_4} \phi \lor \psi$.

• Case $C \Vdash_{C'} \phi ightarrow \psi$. By the induction hypothesis, we know that $C_3 \Vdash_{C_4} \phi$ whenever $C_1 \Vdash_{C_2} \phi$ and $C_3 \Vdash_{C_4} \psi$ whenever $C_1 \Vdash_{C_2} \psi$. Hence we have that $C_3 \Vdash_{C_4} \phi \rightarrow \psi$.

• Case $C_1 \Vdash_{C_2} \langle a \rangle \phi$. As there exist $C'_1$, $C'_2$, and $b$ such that $C_1 \xrightarrow{e_1}^a C'_1$, $C_2 \xrightarrow{c_1}^b C'_2$ and $C'_1 \Vdash_{C'_2} \phi$, by the definition of bisimulation, we know that there exist $C'_3$ and $C'_4$ such that $C_3 \xrightarrow{e_1}^a C'_3$, $C_4 \xrightarrow{c_1}^b C'_4$. By the induction hypothesis, we know that, as $C'_1 \Vdash_{C'_2} \phi$ then $C'_3 \Vdash_{C'_4} \phi$. Hence we have that $C_3 \Vdash_{C_4} \langle a \rangle \phi$.

• Case $C_1 \Vdash_{C_2} [a] \phi$. We have that, for all $C'_1$, $C'_2$, $b$ such that $C_1 \xrightarrow{e_1}^a C'_1$, $C_2 \xrightarrow{c_1}^b C'_2$, $C'_1 \Vdash_{C'_2} \phi$. By the definition of bisimulation, we know that, for all $C'_3$ and $C'_4$ such that $C_3 \xrightarrow{e_1}^a C'_3$, $C_4 \xrightarrow{c_1}^b C'_4$, $C'_1 \sim C'_3$ and $C'_2 \sim C'_4$. By the induction hypothesis, we know that, as $C'_1 \Vdash_{C'_2} \phi$ then $C'_3 \Vdash_{C'_4} \phi$. Hence we have that $C_3 \Vdash_{C_4} [a] \phi$.

• Case $R, E \Vdash_{C_2} I$. Let $C_3 = R, F$. By the hypothesis, we have that $R \equiv e$. By Theorem 1, we have that, as $E \sim I$ and $E \sim F$, $E \sim I$. Hence we have that $R, F \Vdash_{C_4} I$.

• Case $R \circ S, E \Vdash_{C_2} \phi \ast \psi$. Let $C_3 = R, H$. By Theorem 1, as $E \sim H$ and $E \sim F \times G$, we have that $H \sim F \times G$. By the hypothesis, we have that $R, F \Vdash_{C_2(S, [\,] \times G)} \phi$ and $S, G \Vdash_{C_2(S,F \times [\,])} \psi$. By Proposition 1, we have that $C_2(S, [\,] \times G) \sim C_4(S, [\,] \times G)$ and $C_2(S,F \times [\,]) \sim C_4(S,F \times [\,])$. Then, by the induction hypothesis, we then have that $R, F \Vdash_{C_4(S,F \times [\,])} \phi$ and $S, G \Vdash_{C_4(S,F \times [\,])} \psi$, and hence that $R \circ S, H \Vdash_{C_4} \phi \ast \psi$.

• Case $R, E \Vdash_{C_2} \phi \rightarrow \psi$. Let $C_3 = R, G$. By the induction hypothesis, we have that if $S, F \Vdash_{C_2} \phi$, then $S, F \Vdash_{C_4} \phi$, and that $R \circ S, G \times F \Vdash_{C_4} \psi$. Hence we have that $R, G \Vdash_{C_4} \phi \rightarrow \psi$.

□

In the proof of the first part of the Hennessy–Milner property for global bisimulation (Theorem 2) we make explicit use of the fact that global bisimulation is a congruence. We also make implicit use of the sub-property of congruence that global bisimulation is an equivalence relation.

We cannot obtain the first part of the Hennessy–Milner property for the fragment of the logic whose interpretation uses the operators over which local bisimulation is not a congruence (namely multiplicative implication). In order to obtain the first part of the Hennessy–Milner property for the remainder of the logic, it is necessary, however, to show that local bisimulation is an equivalence relation.

Lemma 1. The local equivalence $\approx$ is an equivalence relation; that is, it is reflexive, symmetric, and transitive.
Proof. By induction on the structure of bisimilarities. We give illustrative cases.

- Reflexive. Let $\mathcal{R} = \{((C_1, A, D_1), (C_1, A, D_1)) \mid C_1 : OCont, A : Cont, D_1 : CCont\}$. The relation $\mathcal{R}$ is a local bisimulation if and only if, for all $(C_1, A, D_1) \mathcal{R} (C_2, B, D_2)$, the following holds: for all $C_1' : OCont, A' : Cont, D_1' : CCont$, $a, c, d \in \text{Act}$, if $A \xrightarrow{D_1 a} A'$, $C_1 \xrightarrow{A(D_1)c} C_1'$, and $D_1 \xrightarrow{e_1} d' D_1'$, then there exist $C_2' : OCont, B' : Cont, D_2' : CCont$ such that $B \xrightarrow{D_2 a} B'$, $C_2 \xrightarrow{B(D_2)c} C_2'$, $D_2 \xrightarrow{e_1} d' D_2'$, and, for all $C_1' : OCont, A' : Cont, D_1' : CCont$ such that $A \xrightarrow{D_1 a} A', C_1 \xrightarrow{A(D_1)c} C_1'$, $D_1 \xrightarrow{e_1} d' D_1'$, and $C_1', A', D_1' \approx C_2', B', D_2'$, and $R = S$.

Consider some $(C_1, A, D_1) \mathcal{R} (C_2, B, D_2)$. By the definition of the relation, we know that $C_2 = C_1$, $A = B$, and $D_2 = D_1$. As any evolution of each of the processes $A$, $C_1$, and $D_1$ is decorated with the same outer and inner contexts ($C_1$ and $D_1$, $C_0$ and $A(D_1)$, and $C_1(A)$ and $e_1$, respectively), we immediately have reflexivity.

- Symmetric. Let $\mathcal{R} = \{((C_2, B, D_2), (C_1, A, D_1)) \mid C_1, A, D_1 \approx C_2, B, D_2\}$. If $B \xrightarrow{D_2 a} B'$, $C_2 \xrightarrow{B(D_2)c} C_2'$, and $D_2 \xrightarrow{e_1} d' D_2'$, then we need to show that $A \xrightarrow{D_1 a} A', C_1 \xrightarrow{A(D_1)c} C_1'$, $D_1 \xrightarrow{e_1} d' D_1'$, where $C_1', A', D_1' \approx C_2', B', D_2'$. As $C_1, A, D_1 \approx C_2, B, D_2$, by the definition of local bisimulation, we have that $A \xrightarrow{D_1 a} A', C_1 \xrightarrow{A(D_1)c} C_1'$, $D_1 \xrightarrow{e_1} d' D_1'$, where $C_1', A', D_1' \approx C_2', B', D_2'$. The other case is similar. Hence $\mathcal{R}$ is closed and a bisimulation.

- Transitive. Let $\mathcal{R} = \{((C_2, A_1, D_1), (C_3, A_3, D_3)) \mid C_1, A_1, D_1 \approx C_2, A_2, D_2\}$. If $A_1 \xrightarrow{D_1 a} A_1'$, $C_1 \xrightarrow{A(D_1)c} C_1'$, and $D_1 \xrightarrow{e_1} d' D_1'$, then we need to show that $A_3 \xrightarrow{D_1 a} A_3', C_3 \xrightarrow{A_3(D_1)c} C_3'$, and $D_3 \xrightarrow{e_1} d' D_3'$, where $C_1', A_1', D_1' \mathcal{R} C_3', A_3', D_3'$. By the definition of local bisimulation, as $A_1 \xrightarrow{D_1 a} A_1', C_1 \xrightarrow{A(D_1)c} C_1'$, and $D_1 \xrightarrow{e_1} d' D_1'$, we have that $A_2 \xrightarrow{D_2 a} A_2', C_2 \xrightarrow{A_2(D_2)c} C_2'$, and $D_2 \xrightarrow{e_1} d' D_2'$, where $C_1', A_1', D_1' \approx C_2', A_2', D_2'$. Similarly, as $A_2 \xrightarrow{D_2 a} A_2', C_2 \xrightarrow{A_2(D_2)c} C_2'$, and $D_2 \xrightarrow{e_1} d' D_2'$, we have that $A_3 \xrightarrow{D_3 a} A_3'$, $C_3 \xrightarrow{A_3(D_3)c} C_3'$, and $D_3 \xrightarrow{e_1} d' D_3'$, where $C_2', A_2', D_2' \approx C_3', A_3', D_3'$. As $C_1', A_1', D_1' \approx C_2', A_2', D_2'$ and $C_2', A_2', D_2' \approx C_3', A_3', D_3'$, we have that $C_1', A_1', D_1' \mathcal{R} C_3', A_3', D_3'$. The other case is similar. Hence $\mathcal{R}$ is closed and a bisimulation. \hfill $\Box$
We can then show that the first part of the Hennessy–Milner property holds for local equivalence, with the fragment of the logic that excludes multiplicative implication. Note that we also alter the valuation function so that it maps each atomic proposition to a \( \approx \)-closed set of closed contexts (i.e. if \( C_1, D_1 \approx C_2, D_2 \) and \( C_1(D_1) \in \mathcal{V} \), then \( C_2(D_2) \in \mathcal{V} \).

**Theorem 3.** If \( C_1, A, D_1 \approx C_2, B, D_2 \), then \( C_1, A(D_1) \equiv C_2, B(D_2) \).

**Proof.** Straightforward, by induction over the definition of \( D \equiv C \phi \). This follows essentially as that for Theorem 2. The different cases are as follows.

- Case \( A(D_1) \equiv C_1 \phi \). This is valid if and only if \( C_1(A(D_1)) \in \mathcal{V}(p) \). As \( \mathcal{V} \) is \( \approx \)-closed and \( C_1, A, D_1 \approx C_2, B, D_2 \), we hence have that \( C_2(B(D_2)) \in \mathcal{V}(p) \).

- Case \( A(D_1) \equiv C_1^{\bot} \). By Lemma 1, we have that as \( \approx \) is transitive, so \( C_2, B(D_2) \approx C_2, (e, 1) \), and hence that \( B(D_2) \equiv C_2, \phi \).

- Case \( A(D_1) \equiv C_1 \phi \ast \psi \). By the hypothesis, we have that there exists some \( S, T \) and \( F, G \) such that \( C_1, A(D_1) \equiv C_1^{\bot}, (S \circ T, F \times G) \), and \( S, F \equiv C_2 \phi \) and \( T, G \equiv C_3 \psi \), where \( C_2 = C^2(T, [\,] \times G) \) and \( C_3 = C^2(S, F \times [\,]) \). By Lemma 1, we have that \( \approx \) is transitive, by which we know that \( C_2, B(D_2) \approx C_1, (S \circ T, F \times G) \). Hence, we have \( B(D_2) \equiv C_2, \phi \ast \psi \).

\[ \square \]

We can also obtain a converse, for the local equivalence relation. Define a context to be *image finite* if it has finitely many immediate derivatives (for any given inner and outer contexts with which it reduces). We then have the following.

**Theorem 4.** Consider the modal logic without the multiplicative implication, \( \neg \). If \( C_1, A \equiv C_2, B \), then there exist some \( A_1, D_1, B_1, D_2 \) such that \( A = A_1(D_1), B = B_1(D_2) \), and \( C_1, A_1, D_1 \approx C_2, B_1, D_2 \).

**Proof.** Suppose, for a contradiction, that the theorem is false. Then there must be some contexts \( C_1, A, C_2, B \), with \( C_1, A \equiv C_2, B \) and, without loss of generality, for all \( A_1(D_1) = A \) and \( B_1(D_2) = B \), there exists some transition \( C_1 \xrightarrow{A_1} C_1' \), and transition \( A_1 \xrightarrow{D_1} A_1' \), and transition \( D_1 \xrightarrow{e_1} D_1' \), for some \( C_1', A', \) and \( D_1' \), such that there is no \( C_2', B_1', D_2' \), where \( C_2 \xrightarrow{B_1} C_2', B_1 \xrightarrow{D_2} B_1', D_2 \xrightarrow{e_1} B_1 \). Therefore \( C_1' \approx C_2' \approx C_1', A' \approx C_2', D_2' \approx C_2' \).

Let \( A_1 = A, D_1 = e, 1, B_1 = B, D_2 = e, 1 \). Let \( \mathcal{B} = \{(C_2, B') \mid C_2 \xrightarrow{B} C_2' \) and \( B \xrightarrow{e_1} B'\} \).

If \( \mathcal{B} = \emptyset \) then know that \( A \) can do a \( b \) action and \( B \) cannot, and we can hence show that \( A \equiv C_1 \langle \mathcal{B} \rangle \) and \( B \equiv C_2 \langle \mathcal{B} \rangle \), which contradicts the hypothesis that \( C_1, A \equiv C_2, B \). Therefore \( \mathcal{B} \) must be non-empty. Since \( B \) is image finite (in context \( C_2 \) then we may enumerate the \( n \) elements as \( (C_2', B_1'), \ldots, (C_2', B_n') \). Also, as, (for each \( i \in 1, \ldots, n \)) \( C_1', D_1' \) \( \not\equiv C_2', D_2' \), then there is some \( \phi_i \) such that \( A' \equiv C_1 \phi_i \) and \( B' \equiv C_2 \phi_i \). But then we can show that \( A \equiv C_1 \langle \phi_1 \wedge \ldots \wedge \phi_n \rangle \) and \( B \equiv C_2 \langle \phi_1 \wedge \ldots \wedge \phi_n \rangle \), which contradicts the hypothesis that \( C_1, A \equiv C_2, B \). Therefore \( \mathcal{B} \) cannot be non-empty. As \( \mathcal{B} \) cannot be both empty and non-empty, our supposition must be false and we are done.

\[ \square \]
2 Proofs for ‘Trust Domains: An Algebraic, Logical, and Utility-theoretic Approach’

We now give the technical results that are required for our arguments.

A key technical property, which underpins much of the remaining work, is, if bisimilar contexts are substituted into each other, then the result is bisimilar. For this we provide a proof sketch that we believe suitably underpins the result.

**Proposition 1** (Bisimulation Closure Under Substitution). If \( E \sim G \) and \( F \sim H \), then \( E(F) \sim G(H) \).

**Proof.** The bisimulation relation \( \sim \) is the largest bisimulation relation, and contains all other bisimulation relations. In order to show that \( E(F) \sim G(H) \) it is sufficient, therefore, to define a relation \( \mathcal{R} \), where \( E(F)\mathcal{R}G(H) \), for which the required substitution property holds, and show that the relation \( \mathcal{R} \) is a bisimulation.

Let \( \mathcal{R} = \{(E(F), G(H)) \mid E \sim G \text{ and } F \sim H\} \cup \sim \). The relation is a bisimulation if and only if the following holds: for all \( T, T', I', C_1 \sim C_3, \) and \( C_2 \sim C_4, \) if \( T, E(F) \xrightarrow{C_2} C_1 \rightarrow T', I' \) (respectively \( T, E(F) \xrightarrow{C_2} C_1 \rightarrow T, I' \)), then there exists some \( J' \) such that \( T, G(H) \xrightarrow{C_2} C_1 \rightarrow T', J' \) (respectively \( T, G(H) \xrightarrow{C_2} C_1 \rightarrow T, J' \)), where \( (I', J') \in \mathcal{R} \); and, for all \( T, T', J', C_1 \sim C_3, \) and \( C_2 \sim C_4, \) if \( T, G(H) \xrightarrow{C_2} C_1 \rightarrow T', J' \), \( T, E(F) \xrightarrow{C_2} C_1 \rightarrow T', I' \) (respectively \( T, G(H) \xrightarrow{C_2} C_1 \rightarrow T, J' \)), then there exists some \( I' \) such that \( T, E(F) \xrightarrow{C_2} C_1 \rightarrow T', I' \), where \( (I', J') \in \mathcal{R} \).

All processes are defined by a finite number of applications of the operators of the language. We proceed by induction on the derivation of this structure according to the rules of the operational semantics.

Consider the case in which \( T, E(F) \xrightarrow{C_2} C_1 \rightarrow T, I' \). We prove that there exists some \( J' \) such that \( T, G(H) \xrightarrow{C_2} C_1 \rightarrow T, J' \) by induction on the structures of \( E, G, F, \) and \( H \), and over the (process) structures of \( C_1, C_3, C_2, \) and \( C_4, \) in that order. Here the induction is on the number of operators in a process term.

Consider the case of this nested induction in which \( E = 1, G = G_1 \times G_2, F = 1, \) and \( H = 1, \) where \( C_1 = e, \), \( C_3 = e, \), \( C_2 = e, \), and \( C_4 = e, 1. \)

In SCCS, if \( 1 \sim G_1 \times G_2 \), then \( G_1 \) and \( G_2 \) would necessarily be bisimilar to \( 1, \) but here that is not the case. Consider the process \( G_1 = 1 +_u 1, \) where

\[ u(C) = \begin{cases} 0 & \text{if there exists } C' : OCont, R \text{ where } C = C'(R, G_2) \\ n & \text{otherwise,} \end{cases} \]

for some \( n > 0. \)

We then have that \( (1 +_u 1) \times G_2 \sim 1; \) a sketch of the proof follows below.

Consider some contexts \( C_5 \sim C_7 \) and \( C_6 \sim C_8, \) and resources \( R = R_1 \circ R_2. \) We then have that \( u(C_5(R, 1 \times G_2)) = 0, \) for all \( C_5, \) and hence by the \((S\circ W)\) rule, that \( R_1, 1 +_u 1 \xrightarrow{C_2} C_1 \rightarrow R_1, 1. \) Note, however, that \( 1 +_u 1 \) is not bisimilar to \( 1; \) for the empty outer context \( e, [ ] \) the former can perform a \( n\)-cost transition while the latter cannot.

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As a result, bisimulation does not work component-by-component (solely because of cost-moderated choice). Note, however, that following a cost transition by $1 \rightarrow u \ 1$, the resulting process $1$ is bisimilar to $1$. As the reduction of cost-moderated choices applies to strictly simpler terms, we can apply the induction hypothesis and show that substituted processes are bisimilar, and hence the choices that are formed from them are also bisimilar.

This argument extends to more complex cases, where the subcomponents of a product are not merely tick processes. There, cost moderated choices as subcomponents will also eventually reduce to a product subcomponent to which the induction hypothesis can be applied.

The proof sketched above makes use of an eight-fold nested induction, which has a very large number of cases. Unfortunately, the complexity of this nested induction precludes an exhaustive presentation. However, it is easy to see that the difficult cases in the induction arise solely from use of cost-moderated choices and, since these always reduce to simpler processes to which the induction hypothesis can be applied, the key cases are all similar to the one described above.

The remaining cases are routine. 

Note that the argument presented above would not be applicable in the presence of general (guarded) fixed points; our contextual calculus must be a finite modelling framework.

With this result, we can obtain a key property for reasoning compositionally, that bisimulation is a congruence.

**Theorem 1** (Bisimulation Congruence). The relation $\sim$ is a congruence. It is reflexive, symmetric, and transitive, and, for all $a, E, F, G$, with $E \sim F$ and all families $(E_i)_{i \in I}, (F_i)_{i \in I}$ with $E_i \sim F_i$, for all $i \in I$, $a : E \sim a : F$, $E \times G \sim F \times G$, and $\sum_{i \in I} E_i \sim \sum_{i \in I} F_i$.

**Proof.** By induction on the structure of bisimulations. We give illustrative cases.

1. Reflexive. Let $R = \{(E, E)\}$. As the evolution of each of the process $E$ is decorated with different outer and inner contexts ($C_1$ and $C_2$, and $C_3$ and $C_4$, respectively) we do not immediately have reflexivity. We prove this property by induction on the derivation of $R, E \xrightarrow{C_1} C_2^n R, E'$. The relation $R$ is a bisimulation if and only if, for all $E$ and $F$ such that $E \mathcal{R} F$, the following holds: for all $R, R', E', a, n, C_1 \sim C_3$, and $C_2 \sim C_4$, if $R, E \xrightarrow{C_1} C_2^n R', E'$ (respectively $R, E \xrightarrow{C_1} C_2^n R, F'$), then there exists some $F'$ such that $R, F \xrightarrow{C_3} R', F'$ (respectively $R, F \xrightarrow{C_3} R, F'$), where $E' \mathcal{R} F'$; and, for all $R, R', F', a, n, C_1 \sim C_3$, and $C_2 \sim C_4$, if $R, F \xrightarrow{C_3} R', F'$ (respectively $R, F \xrightarrow{C_3} R, F'$), then there exists some $E'$ such that $R, E \xrightarrow{C_1} C_2^n R', E'$ (respectively $R, E \xrightarrow{C_1} C_2^n R, E'$), where $E' \mathcal{R} F'$.

- Consider some $E \mathcal{R} F, R, R', E', a, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, E \xrightarrow{C_1} C_2^n R', E'$. By the definition of $R$, we know that $F = E$. We then prove that there exists some $E''$, such that $R, E \xrightarrow{C_1} C_2^n R', E''$ and $E' \mathcal{R} E''$, by induction over the derivation of $R, E \xrightarrow{C_1} C_2^n R', E'$.

- Consider some $E \mathcal{R} F, R, R', E', n, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, E \xrightarrow{C_1} C_2^n R, E'$. By the definition of $R$, we know that $F = E$. We then prove that there exists some $E''$, such that $R, E \xrightarrow{C_1} C_2^n R, E''$ and $E' \mathcal{R} E''$ by induction over the derivation of $R, E \xrightarrow{C_1} C_2^n R, E'$.
5. Let $R \triangleq F, R', F', a, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, F \overset{C_4}{\rightarrow}_{C_1} R', F'$. By the definition of $\mathcal{R}$, we know that $F = E$ and $F' = E''$. We then prove that there exists some $E'$, such that $R, E \overset{C_4}{\rightarrow}_{C_1} R', E'$ and $E' \mathcal{R} E''$, by induction over the derivation of $R, E \overset{C_4}{\rightarrow}_{C_1} R', E''$.

6. Consider some $E \mathcal{R} F, R, R', F', n, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, F \overset{C_4}{\rightarrow}_{C_1} R', F'$. By the definition of $\mathcal{R}$, we know that $F = E$ and $F' = E''$. We then prove that there exists some $E'$, such that $R, E \overset{C_4}{\rightarrow}_{C_1} R', E'$ and $E' \mathcal{R} E''$, by induction over the derivation of $R, E \overset{C_4}{\rightarrow}_{C_1} R', E''$.

Hence $\mathcal{R}$ is closed and a bisimulation.

2. Symmetric. Let $\mathcal{R} = \{(F, E) \mid E \sim F\} \cup \sim$. If $R, F \overset{C_4}{\rightarrow}_{C_1} S, F', C_1 \sim C_3$, and $C_2 \sim C_4$, then we need to show that $R, E \overset{C_4}{\rightarrow}_{C_1} S', E'$, where $F' \mathcal{R} E'$. As $E \sim F$, by the definition of bisimulation, we have that, if $C_5 \sim C_7, C_6 \sim C_8$, and $R, F \overset{C_4}{\rightarrow}_{C_1} S, F'$, then $R, E \overset{C_4}{\rightarrow}_{C_1} S, E'$, where $E' \sim F'$.

Let $C_1 = C_5, C_2 = C_6, C_3 = C_7$, and $C_4 = C_8$. We then have that $R, E \overset{C_4}{\rightarrow}_{C_1} S, E'$. As $E' \sim F'$ we have that $F' \mathcal{R} E'$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.

3. Transitive. Let $\mathcal{R} = \{(E, G) \mid E \sim F \text{ and } F \sim G\}$. If $R, E \overset{C_4}{\rightarrow}_{C_1} S, E', C_1 \sim C_3$, and $C_2 \sim C_4$, then we need to show that $R, G \overset{C_4}{\rightarrow}_{C_1} S, G'$, where $E' \mathcal{R} G'$. By the definition of bisimulation, as $R, E \overset{C_4}{\rightarrow}_{C_1} S, E'$, we have that $R, F \overset{C_4}{\rightarrow}_{C_1} S, F'$, where $E' \sim F'$, and similarly, as $R, F \overset{C_4}{\rightarrow}_{C_1} S, F'$, we have that $R, G \overset{C_4}{\rightarrow}_{C_1} S, G'$, where $F' \sim G'$. We then have that $E' \mathcal{R} G'$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.

4. Let $\mathcal{R} = \{(a : E, a : F) \mid E \sim F\} \cup \sim$. If $R, a : E \overset{C_4}{\rightarrow}_{C_1} S, E, C_1 \sim C_3$, and $C_2 \sim C_4$, then we need to prove that $R, a : F \overset{C_4}{\rightarrow}_{C_1} S, F$, where $E \mathcal{R} F$. The only applicable reduction rule for $a : E$ is the (Prefix) rule. By this rule, which disregards $C_3$ and $C_4$, we can show that $R, a : F \overset{C_4}{\rightarrow}_{C_1} S, F$. As $E \sim F$, we have that $E \mathcal{R} F$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.

5. Let $\mathcal{R} = \{(E_+ u G, F_+ u G) \mid E \sim F\} \cup \sim$. If $R, E_+ u G \overset{C_4}{\rightarrow}_{C_1} R, E', C_1 \sim C_3$, and $C_2 \sim C_4$, then we need to prove that $R, F_+ u G \overset{C_4}{\rightarrow}_{C_1} R, F'$ and $E' \mathcal{R} F'$. By (SimW), we know that the that $n = u(C_1(R, E(C_2)))$, and $E' = E$. As $E \sim F$ and $C_5 \sim C_6$ (by Proposition 1), by Definition 1, we have that $u(C_3(R, F(C_4))) = u(C_1(R, E(C_2)))$. Then, by (SimW), we can show that $R, F_+ u G \overset{C_4}{\rightarrow}_{C_1} S, F$. As $E \sim F$, we have that $E \mathcal{R} F$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.
6. Let $\mathcal{R} = \{(E \times G, F \times G) \mid E \sim F\}$. If $R \circ S, E \times G \xrightarrow{C_3 \cdot C_1} R' \circ S', F' \times G'$, we need to prove that $R \circ S, F \times G \xrightarrow{C_2 \cdot C_1} R' \circ S', F' \times G''$. By the \(\text{(Proo)}\) rule, we have that $R, E \xrightarrow{C_4 \cdot C_1} R', E'$ and $S, G \xrightarrow{C_2 \cdot C_1} S', G'$, where $C_5 = C_1((S, F(C_2)) \times [\cdot])$ and $C_6 = C_1((R, E(C_2)) \times [\cdot])$. By Proposition 1 and $\mathcal{R}$, we have that $C_5 \sim C_7 = (C_3(S, G(C_4)) \times [\cdot]$ and $C_6 \sim C_8 = (C_5(R, F(C_4)) \times [\cdot]$. By the definition of bisimulation, we have that $R, F \xrightarrow{C_4 \cdot C_1} R', F'$ and $S, G \xrightarrow{C_2 \cdot C_1} S', G''$, where $E' \sim F'$ and $G' \sim G''$. We can then use the \(\text{(Proo)}\) rule to show that $R \circ S, F \times G \xrightarrow{C_2 \cdot C_1} R' \circ S', F' \times G''$, where $E' \times G' \Rightarrow F' \times G''$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.

\[\square\]

In order to reason equationally about processes, it is also useful to establish various algebraic properties concerning parallel composition and choice. We derive these properties for our calculus below. We use the binary version of sum here in order to aid comprehension, but finite choices between sets of processes work straightforwardly.

**Proposition 2** (Algebraic Properties). For all $a, E, F, G$, we have the following: (1) $E +_u F \sim F +_u E$; (2) $E \times 0 \sim 0$; (3) $E \times 1 \sim E$; (4) $E \times F \sim F \times E$; and (5) $E \times (F \times G) \sim (E \times F) \times G$.

**Proof.**

1. Let $\mathcal{R} = \{(E +_u F, F +_u E) \mid E, F : \text{PCont}\} \cup \sim$. The relation $\mathcal{R}$ is a bisimulation if and only if, for all $E$ and $F$ such that $E \not\sim F$, the following holds: for all $R, R'$, $E, a, n, C_1 \sim C_3$, and $C_2 \sim C_4$, if $R, E \xrightarrow{C_2 \cdot C_1} R', E'$ (respectively $R, E \xrightarrow{C_3 \cdot C_1} R, E'$), then there exists some $F'$ such that $R, F \xrightarrow{C_4 \cdot C_1} R', F'$ (respectively $R, F \xrightarrow{C_4 \cdot C_1} R, F'$), where $E' \Rightarrow F'$. Then, for all $R, R'$, $F, a, n, C_1 \sim C_3$, and $C_2 \sim C_4$, if $R, F \xrightarrow{C_2 \cdot C_1} R', F'$ (respectively $R, F \xrightarrow{C_3 \cdot C_1} R, F'$), then there exists some $E'$ such that $R, E \xrightarrow{C_2 \cdot C_1} R', E'$ (respectively $R, E \xrightarrow{C_3 \cdot C_1} R, E'$), where $E' \Rightarrow F'$.

- Consider some $E_1 \mathcal{R} E_2, R, R', E'_1, a, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, E_1 \xrightarrow{C_2 \cdot C_1} R', E'_1$. Consider the case in which $E_1 \sim E_2$. Then, by the definition of bisimulation, we have that there exists some $E'_2$ such that $E_2 \xrightarrow{C_3 \cdot C_1} R', E'_2$, where $E'_1 \sim E'_2$, and hence $E'_1 \mathcal{R} E'_2$. Consider the case in which $E_1 = E +_u F$ and $E_2 = F +_u E$. There is no action reduction rule for the sum operator, and therefore there are no such transitions $R, E_1 \xrightarrow{C_2 \cdot C_1} R', E'_1$. As such, this case is vacuously true.

- Consider some $E_1 \mathcal{R} E_2, R, R', E'_1, n, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, E_1 \xrightarrow{C_3 \cdot C_1} R', E'_1$. Consider the case in which $E_1 \sim E_2$. Then, by the definition of bisimulation, we have that there exists some $E'_2$ such that $E_2 \xrightarrow{C_4 \cdot C_1} R', E'_2$, where $E'_1 \sim E'_2$, and hence $E'_1 \mathcal{R} E'_2$. Consider the case in which $E_1 = E +_u F$ and $E_2 = F +_u E$. By the \(\text{(SomW)}\) rule, we have that either $n = u(C_1(R, E(C_2)))$ and $E'_1 = E$, or $n = u(C_1(R, F(C_2)))$ and $E'_1 = F$. Consider
3. Let $R \sim C_2$ and, by Definition 1, we have that $n = u(C(R, E(C_2)))$. Then, by the (SumW) rule, we can derive $R, F +_u E \xrightarrow{C_n} R, E$.

- Consider some $E_1 \in \mathcal{R} \mathcal{E}_2, R, R', E_2', a, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, E_2 \xrightarrow{C_2} R', E_2'$. Consider the case where $E_1 \sim E_2$. Then, by the definition of bisimulation, we have that there exists some $E_1'$ such that $R, E_1 \xrightarrow{C_2} R', E_1'$, where $E_1' \sim E_2'$, and hence $E_1' \in \mathcal{R} \mathcal{E}_2'$. Consider the case in which $E_1 = E +_u F$ and $E_2 = F +_u E$. There is no action reduction rule for the sum operator, and therefore there are no such transitions $R, E_2 \xrightarrow{C_2} R', E_2'$. As such, this case is vacuously true.

- Consider some $E_1 \in \mathcal{R} \mathcal{E}_2, R, R', E_2', a, C_1 \sim C_3$, and $C_2 \sim C_4$ such that $R, E_w \xrightarrow{C_n} R', E_2'$.

Consider the case in which $E_1 \sim E_2$. Then, by the definition of bisimulation, we have that there exists some $E_1'$ such that $R, E_1 \xrightarrow{C_2} R', E_1'$, where $E_1' \sim E_2'$, and hence $E_1' \in \mathcal{R} \mathcal{E}_2'$. Consider the case in which $E_1 = E +_u F$ and $E_2 = F +_u E$. By the (SumW) rule, we have that either $n = u(C_3(R, F(C_4)))$ and $E_2' = F$, or $n = u(C_3(R, E(C_4)))$ and $E_2' = E$. Consider the former case (the other is symmetric). By Proposition 1, we have that $C_1((R, F(C_2))) \sim C_3(R, F(C_4))$, and by Definition 1, we have that $n = u(C_1(R, F(C_2)))$. Then, by the (SumW) rule, we can derive $R, F +_u E \xrightarrow{C_n} R, E$.

Hence $\mathcal{R}$ is closed and a bisimulation.

2. Let $\mathcal{R} = \{(E \times 0) \mid E \in \mathcal{P}Cont\}$. By the operational semantics, we have that 0 can make neither action nor weighted transitions. The only applicable rules to $E \times 0$ are (Para0) and (ParaW). These both require, as sub-derivations, that 0 make a transition, which is impossible. Hence $E \times 0$ can also make no transitions, and is bisimilar to 0.

3. Let $\mathcal{R} = \{(E \times 1, F) \mid E \sim F\}$. If $R, E \times 1 \xrightarrow{C_2} S, E', C_1 \sim C_3$ and $C_2 \sim C_4$, then we need to show that $R, F \xrightarrow{C_2} S, F'$, where $E' \mathcal{R} F'$. Let $C_5 = C_1((e, 1(C_2)) \times \{\})$. By the (Para) rule, we have that $R, E \xrightarrow{C_2} S, E'$. By $\mathcal{R}$, we have that $\{\} \times 1 \sim \{\}$ and, by Proposition 1, we then have that $C_3 \sim C_5 \sim C_1$. We can then show that $R, E \xrightarrow{C_4} S, F'$, where $E' \sim F'$. We then have that $E' \times 1 \mathcal{R} F'$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.

4. Let $\mathcal{R} = \{(E \times F, F \times E) \mid E, F : \mathcal{P}Cont\}$. If $R \circ S, E \times F \xrightarrow{C_2} R' \circ S', E' \times F'$, $C_1 \sim C_3$ and $C_2 \sim C_4$, then we need to show that $R \circ S, F \times E \xrightarrow{C_2} R' \circ S', F'' \times E''$, where $E' \times F' \mathcal{R} F'' \times F''$. Let $C_5 = C_1((S, F(C_2)) \times \{\})$ and $C_6 = C_1((R, E(C_2)) \times \{\})$. By the (Para) rule, we have that $R, E \xrightarrow{C_2} R', E'$ and $S, F \xrightarrow{C_2} S', F'$. Let $C_7 = C_3((S, F(C_4)) \times \{\})$ and $C_8 = C_3((R, E(C_4)) \times \{\})$. By $\mathcal{R}$ and Proposition 1, we have that $C_5 \sim C_7$ and $C_6 \sim C_8$. By the definition of bisimulation, we have that $R, E \xrightarrow{C_4} R', E''$ and $S, F \xrightarrow{C_4} S', F''$, where $E' \sim E''$. 

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and $F' \sim F''$. Then, by the (Pfood) rule, we have that $R \circ S, F \times E \xrightarrow{C_4} R' \circ S', F' \times E'$, where $(E' \times F', F' \times E') \in \mathcal{R}$. As $E' \sim E''$ and $F' \sim F''$, we have that $E' \times F' \not\sim F'' \times E''$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.

5. Let $\mathcal{R} = \{(E \times (F \times G), (E \times F) \times G) \mid E, F, G : \text{PCont}\}$. If $R \circ S \circ T, E \times (F \times G) \xrightarrow{C_1} R' \circ S' \circ T', (E' \times F') \times G'$, then we need to show that $R \circ S \circ T, (E \times F) \times G \xrightarrow{C_1} R' \circ S' \circ T', (E'' \times F') \times G''$, where $E' \times (F' \times G') \not\sim (E'' \times F'') \times G''$. Let $C_5 = C_1((S \circ T, F \times G(C_2)) \times \emptyset$) and $C_6 = C_1((R, E(C_2)) \times \emptyset$. By the (Pfood) rule, we have that $R, E \xrightarrow{C_2} R', E'$ and that $S \circ T, F \times G \xrightarrow{C_2} S' \circ T', F' \times G'$. Let $C_7 = C_5((T, G(C_2)) \times \emptyset$) and $C_8 = C_6((S, F(C_2)) \times \emptyset$. By the (Pfood) rule, we have that $S, F \xrightarrow{C_3} S', F'$ and that $T, G \xrightarrow{C_8} T', G'$.

Now, $C_9 = C_3(T, G(C_4)) \times \emptyset, C_{10} = C_9((S, F(C_4)) \times \emptyset, C_{11} = C_9((R, E(C_4)) \times \emptyset, and $C_{12} = C_3((R \circ S, E \times(F(C_4)) \times \emptyset$).

By $\mathcal{R}$ and Proposition 1, we have that $C_8 \sim C_{12}, C_7 \sim C_{11},$ and $C_5 \sim C_{10}$. By the definition of bisimulation, we have that $R, E \xrightarrow{C_4} R', E'$, $S, F \xrightarrow{C_4} S', F'$, and $T, G \xrightarrow{C_4} T', G'$, where $E' \sim E''$, $F' \sim F''$, and $G' \sim G''$. Then, by the (Pfood) rule, we have that $R \circ S, E \times F \xrightarrow{C_4} R' \circ S', E'' \times F''$, and that $R \circ S \circ T, (E \times F) \times G \xrightarrow{C_4} R' \circ S' \circ T', (E'' \times F'') \times G''$. As $E' \sim E''$, $F' \sim F''$, and $G' \sim G''$, we have that $E' \times (F' \times G') \not\sim (E'' \times F'') \times G''$. The other cases are similar. Hence $\mathcal{R}$ is closed and a bisimulation.

Behaviourally equivalent processes are also logically equivalent (they satisfy the same logical properties).

**Theorem 2.** If $C_1 \models_{C_2} \phi$, and $C_1 \sim C_3$, and $C_2 \sim C_4$, then $C_3 \models_{C_4} \phi$.

**Proof.** By induction over the derivation of $C_1 \models_{C_2} \phi$.

- **Case $C_1 \models_{C_2} p$.** By the definition of $\mathcal{V}$, we have that, if $C \sim C'$ and $C \in \mathcal{V}(p)$, then $C' \in \mathcal{V}(p)$. By Proposition 1, we have that $C_2(C_1) \sim C_4(C_3)$, and hence $C_4(C_3) \in \mathcal{V}(p)$.

- **Case $C_1 \models_{C_2} \bot$.** As the premisses assume $C_1 \models_{C_2} \phi$, we have a contradiction and can disregard this case.

- **Case $C_1 \models_{C_2} T$.** We have that $C_3 \models_{C_4} T$, straightforwardly.

- **Case $C_1 \models_{C_2} \phi \land \psi$.** By the induction hypothesis, we know that $C_3 \models_{C_4} \phi$ and $C_3 \models_{C_4} \psi$. Hence we have that $C_3 \models_{C_4} \phi \land \psi$.

- **Case $C \models_{C'} \phi \lor \psi$.** By the induction hypothesis, we know that $C_3 \models_{C_4} \phi$ or $C_3 \models_{C_4} \psi$. Hence we have that $C_3 \models_{C_4} \phi \lor \psi$.

- **Case $C \models_{C'} \phi \rightarrow \psi$.** By the induction hypothesis, we know that $C_3 \models_{C_4} \phi$ whenever $C_1 \models_{C_2} \phi$ and $C_3 \models_{C_4} \psi$ whenever $C_1 \models_{C_2} \psi$. Hence we have that $C_3 \models_{C_4} \phi \rightarrow \psi$.

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• Case $C_1 \models_{C_2} \langle \phi \rangle$. As there exist $C_1'$, $C_2'$, and $b$ such that $C_1 \xrightarrow{C_2} C_1'$ and $C_2 \xrightarrow{C_0} C_2'$ and $C_1' \models_{C_2'} \phi$, then, by the definition of bisimulation, we know that there exist $C_3'$ and $C_4'$ such that $C_3 \xrightarrow{C_2} C_3'$ and $C_4 \xrightarrow{C_0} C_4'$. By the induction hypothesis, we know that, as $C_1' \models_{C_2'} \phi$, then $C_3' \models_{C_4'} \phi$. Hence we have that $C_3 \models_{C_4} \langle \phi \rangle$.

• Case $C_1 \models_{C_2} \langle a \rangle \phi$. We have that, for all $C_1'$, $C_2'$, $b$ such that $C_1 \xrightarrow{C_2} C_1'$ and $C_2 \xrightarrow{C_0} C_2'$, $C_1' \models_{C_2'} \phi$. By the definition of bisimulation, we know that, for all $C_3'$ and $C_4'$ such that $C_3 \xrightarrow{C_2} C_3'$ and $C_4 \xrightarrow{C_0} C_4'$, we have that $C_1' \models_{C_2'} \phi$. Hence we have that $C_3 \models_{C_4} \langle a \rangle \phi$.

• Case $R, E \models_{C_2} I$. Let $C_3 = R, F$. By Theorem 1, we have that, as $E \sim 1$ and $E \sim F$, $E \sim 1$. Hence we have that $R, F \models_{C_3} I$.

• Case $R \circ S, E \models_{C_2} \phi \circ \psi$. Let $C_3 = R, H$. By Theorem 1, we have that as $E \sim H$ and $E \sim F \times G$ that hence $H \sim F \times G$. By the induction hypothesis, we have that $R, F \models_{C_2(S, \{1\} \times G)} \phi$ and $S, G \models_{C_2(S, F \times \{1\})} \psi$. By Proposition 1, we have that $C_2(S, \{1\} \times G) \sim C_4(S, \{1\} \times G)$ and $C_2(S, F \times \{1\}) \sim C_4(S, F \times \{1\})$. Then, by the induction hypothesis, we know that, as $R, F \models_{C_2(S, \{1\} \times G)} \phi$ and $S, G \models_{C_2(S, F \times \{1\})} \psi$, hence that $R \circ S, H \models_{C_3} \phi \circ \psi$.

• Case $R, E \models_{C_2} \phi \leadsto \psi$. Let $C_3 = R, G$. By the induction hypothesis, we have that if $S, F \models_{C_2} \phi$, then $S, F \models_{C_4} \phi$ and that $R \circ S, G \times F \models_{C_4} \psi$. Hence we have that $R, G \models_{C_3} \phi \leadsto \psi$.

• Case $C_1 \models_{C_2} \langle \leq n \rangle \phi$. As there exist $C_1'$, $C_2'$, and $m$ such that $C_1 \xrightarrow{C_2} C_1'$ and $C_2 \xrightarrow{C_0} C_2'$, $C_1' \models_{C_2'} \phi$, and $m \leq n$, by the definition of bisimulation, we know that there exist $C_3'$ and $C_4'$ such that $C_3 \xrightarrow{C_2} C_3'$ and $C_4 \xrightarrow{C_0} C_4'$, where $C_1' \sim C_3'$ and $C_2' \sim C_4'$. By the induction hypothesis, we know that, as $C_1' \models_{C_2'} \phi$ then $C_3' \models_{C_4'} \phi$. Hence we have that $C_3 \models_{C_4} \langle \leq n \rangle \phi$.

• Case $C_1 \models_{C_2} \langle \leq n \rangle \phi$. We have that, for all $C_1'$ and $C_2'$ such that $C_1 \xrightarrow{C_2} C_1'$ and $C_2 \xrightarrow{C_0} C_2'$, $C_1' \models_{C_2'} \phi$. By the definition of bisimulation, we know that, for all $C_3'$ and $C_4'$ such that $C_3 \xrightarrow{C_2} C_3'$ and $C_4 \xrightarrow{C_0} C_4'$, $C_1' \sim C_3'$ and $C_2' \sim C_4'$. By the induction hypothesis, we know that as $C_1' \models_{C_2'} \phi$, $C_3' \models_{C_4'} \phi$. Hence we have that $C_3 \models_{C_4} \langle \leq n \rangle \phi$.
that $C_3 \xrightarrow{c_0} m_{C_3}$ and $C_4 \xrightarrow{c_0} o_{C_4}$, where $C_3' \sim C_3$ and $C_4' \sim C_4$. By the induction hypothesis, we know that, as $C_1' \vdash_{C_2'} \phi, C_3' \vdash_{C_4'} \phi$. Hence we have that $C_3 \vdash_{C_4} [a] \phi$. □
References
